# WEAKENING IDEMPOTENCY IN $K$-THEORY

V. MANUILOV

Abstract. We show that the $K$-theory of $C^*$-algebras can be defined by pairs of matrices satisfying less strict relations than idempotency.

## 1. Introduction

$K$-theory of a $C^*$-algebra $A$ is patently defined by pairs (formal differences) of idempotent matrices (projections) over $A$. Regretfully, being a projection is a very strict property, and it is usually very hard to find projections in a given $C^*$-algebra. Many famous conjectures (Kadison, Novikov, Baum–Connes, Bass, etc.) are related to projections and would become more tractable if one could provide enough projections for a given $C^*$-algebra. Our aim is to show that the $K$-theory can be defined using less restrictive relations in hope that it would be easier to find elements satisfying these relations than the genuine idempotency. We show that $K$-theory is generated by pairs $a, b$ of matrices over $A$ satisfying $(a-a^2)(a-b) = (b-b^2)(a-b) = 0$, which means that $a$ and $b$ have to be "projections" only when $a \neq b$.

## 2. Definitions and some properties

Let $A$ be a $C^*$-algebra. For $a, b \in A$, consider the relations

$$\|a\| \leq 1; \quad \|b\| \leq 1; \quad a, b \geq 0; \quad (a-a^2)(a-b) = 0; \quad (b-b^2)(a-b) = 0. \tag{1}$$

Two pairs, $(a_0, b_0)$ and $(a_1, b_1)$ of elements in $A$, are *homotopy equivalent* if there are paths $a = (a_t), b = (b_t) : [0,1] \to A$, connecting $a_0$ with $a_1$ and $b_0$ with $b_1$ respectively, such that the relations

$$\|a_t\| \leq 1; \quad \|b_t\| \leq 1; \quad a_t, b_t \geq 0; \quad (a_t - a_t^2)(a_t - b_t) = 0; \quad (b_t - b_t^2)(a_t - b_t) = 0$$

hold for each $t \in [0,1]$.

A pair $(a, b)$ is *homotopy trivial* if it is homotopy equivalent to $(0, 0)$.

**Lemma 2.1.** *The pair $(a, a)$ is homotopy trivial for any $a \in A$.*

*Proof.* The linear homotopy $a_t = t \cdot a$ would do. ∎

**Lemma 2.2.** *If $a, b$ satisfy (1) then $f(a) = f(b)$ and $f(a)(a-b) = 0$ for any $f \in C_0(0,1)$.*

*Proof.* It follows from $(a-a^2)(a-b) = 0$, or, equivalently, from $(a-a^2)a = (a-a^2)b$, that

$$(a-a^2)a^2 = a(a-a^2)a = a(a-a^2)b = (a-a^2)b^2,$$

hence

$$(a-a^2)(a-a^2) = (a-a^2)(b-b^2).$$

Similarly,

$$(b-b^2)(b-b^2) = (a-a^2)(b-b^2),$$

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therefore

$$(a-a^2)^2 = (b-b^2)^2. \tag{2}$$

Then (2) and positivity of $a-a^2$ and $b-b^2$ imply that

$$a-a^2 = b-b^2.$$

Also,

$$(a-a^2)a = (a-a^2)b = (b-b^2)b.$$

Since the two functions $g, h$, $g(t) = t-t^2$, $h(t) = tg(t)$, generate $C_0(0,1)$, and $g(a) = g(b)$, $h(a) = h(b)$, we conclude that the same holds for any $f \in C_0(0,1)$. Similarly, $g(a)(a-b) = 0$ and $h(a)(a-b) = 0$ implies $f(a)(a-b) = 0$ for any $f \in C_0(0,1)$.

□

**Corollary 2.3.** *If $\|a\| < 1$, $\|b\| < 1$ and the pair $(a,b)$ satisfies (1) then $a = b$, hence the pair $(a,b)$ is homotopy trivial.*

*Proof.* Take $f \in C_0(0,1)$ such that $f(t) = t \in \mathrm{Sp}(a) \cup \mathrm{Sp}(b)$ and $f(1) = 0$. Then $a = f(a)$, $b = f(b)$, and the claim follows from Lemma 2.2.

□

**Lemma 2.4.** *The pair $(f(a), f(b))$ is homotopy equivalent to $(a,b)$ for any continuous map $f : [0,1] \to [0,1]$ such that $f(0) = 0$, $f(1) = 1$.*

*Proof.* As the set of all functions with the stated properties is convex, so it suffices to show that for any such function $f$, the pair $(f(a), f(b))$ satisfies the relations (1).

Set $f_0(t) = f(t) - t$. Then $f_0 \in C_0(0,1)$. As $f_0(a) = f_0(b)$ by Lemma 2.2, so

$$f(a) - f(b) = a - b.$$

Set

$$g(t) = t - t^2 + f_0(t) - f_0^2(t) - 2tf_0(t).$$

Then $g \in C_0(0,1)$ and

$$(f(a) - f^2(a))(f(a) - f(b)) = g(a)(a-b) = 0;$$
$$(f(b) - f^2(b))(f(a) - f(b)) = g(a)(a-b) = 0.$$

□

**Corollary 2.5.** $\mathrm{Sp}(a) \setminus \{0,1\} = \mathrm{Sp}(b) \setminus \{0,1\}$.

*Proof.* The inner points of $[0,1]$ in the two spectra must coinside by Lemma 2.2. □

Let $M_n(A)$ denote the $n \times n$ matrix algebra over $A$. Two pairs, $(a_0, b_0)$ and $(a_1, b_1)$, where $a_0, a_1, b_0, b_1 \in M_n(A)$, are equivalent if there is a homotopy trivial pair $(a,b)$, $a, b \in M_m(A)$ for some integer $m$, such that the pairs $(a_0 \oplus a, b_0 \oplus b)$ and $(a_1 \oplus a, b_1 \oplus b)$ are homotopy equivalent in $M_{n+m}(A)$. Using the standard inclusion $M_n(A) \subset M_{n+k}(A)$ (as the upper left corner) we may speak about equivalence of pairs of different matrix size.

Let $[(a,b)]$ denote the equivalence class of the pair $(a,b)$, $a, b \in M_n(A)$.

For two pairs, $(a,b)$, $a, b \in M_n(A)$, and $(c,d)$, $c, d \in M_m(A)$, set

$$[(a,b)] + [(c,d)] = [(a \oplus c, b \oplus d)].$$

The result obviously doesn't depend on a choice of representatives. Also $[(a,b)] + [(c,d)] = [(a,b)]$ when $(c,d)$ is homotopy trivial.

**Lemma 2.6.** *The addition is commutative and associative.*

*Proof.* If $(u_t)_{t\in[0,1]}$ is a path of unitaries in $A$, $u_1 = 1$, $u_0 = u$, then $[(u^*au, u^*bu)] = [(a,b)]$ for any $a, b \in A$, as the relations (1) are not affected by unitary equivalence. The standard argument with a unitary path connecting $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ proves commutativity. A similar argument proves associativity. ∎

**Lemma 2.7.** $[(a,b)] + [(b,a)] = [(0,0)]$ *for any* $a, b$.

*Proof.* Set $U_t = \left(\begin{smallmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{smallmatrix}\right)$, $B_t = U_t^*\left(\begin{smallmatrix}b&0\\0&a\end{smallmatrix}\right)U_t$. We claim that the pair $\left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right), B_t\right)$ satisfies the relations (1) for all $t$.

One has

$$B_t = \begin{pmatrix} b\cos^2 t + a\sin^2 t & (a-b)\cos t\sin t\\ (a-b)\cos t\sin t & b\sin^2 t + a\cos^2 t\end{pmatrix} = \left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) + (a-b)C_t, \tag{3}$$

where $C_t = \left(\begin{smallmatrix}-\cos^2 t & \cos t\sin t\\ \cos t\sin t & \cos^2 t\end{smallmatrix}\right)$.

Then

$$\left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) - \left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)^2\right)\left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) - B_t\right) = \begin{pmatrix} a-a^2 & 0\\ 0 & b-b^2\end{pmatrix}(a-b)C_t = \begin{pmatrix}(a-a^2)(a-b) & 0\\ 0 & (b-b^2)(a-b)\end{pmatrix}C_t = 0.$$

It remains to show that

$$A = (B_t - B_t^2)\left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) - B_t\right) = 0.$$

Using (3) we have

$$\begin{aligned}
A &= \left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) + (a-b)C_t - \left(\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) + (a-b)C_t\right)^2\right)(a-b)C_t\\
&= \left(\left(\begin{smallmatrix}a-a^2&0\\0&b-b^2\end{smallmatrix}\right) + (a-b)C_t - \left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)(a-b)C_t - C_t(a-b)\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) - (a-b)^2C_t^2\right)(a-b)C_t\\
&= \left((a-b)C_t - \left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right)(a-b)C_t - C_t(a-b)\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) - (a-b)^2C_t^2\right)(a-b)C_t\\
&= \left(\left(\begin{smallmatrix}a-b-a^2+ab&0\\0&a-b-ba+b^2\end{smallmatrix}\right)C_t - C_t(a-b)\left(\begin{smallmatrix}a&0\\0&b\end{smallmatrix}\right) - (a-b)^2\cos^2 t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)(a-b)C_t\\
&= \left(\left(\begin{smallmatrix}-b+ab&0\\0&a-ba\end{smallmatrix}\right)C_t - C_t\left(\begin{smallmatrix}a-ba&0\\0&ab-b\end{smallmatrix}\right) - (a-b)^2\cos^2 t\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)(a-b)C_t\\
&= \left(\left(\begin{smallmatrix}(ab+ba-a-b)\cos^2 t&0\\0&(ab+ba-a-b)\cos^2 t\end{smallmatrix}\right) - \left(\begin{smallmatrix}(a-b)^2\cos^2 t&0\\0&(a-b)^2\cos^2 t\end{smallmatrix}\right)\right)(a-b)C_t = 0.
\end{aligned}$$

Thus, the pair $(a\oplus b, b\oplus a)$ is homotopy equivalent to the pair $(a\oplus b, a\oplus b)$, and the latter is homotopy trivial by Lemma 2.1. ∎

So we see that the equivalence classes of pairs satisfying the relations (1) in matrix algebras over $A$ form an abelian group for any $C^*$-algebra $A$. Let us denote this group by $L(A)$.

Note that pairs of projections patently satisfy the relations (1). If $A$ is a unital $C^*$-algebra then $K_0(A)$ consists of formal differences $[p]-[q]$ with $p, q$ projections in matrices over $A$. Then

$$\iota([p]-[q]) = [(p,q)]$$

gives rise to a morphism $\iota : K_0(A) \to L(A)$.

In the non-unital case, $\iota$ can be defined after unitalization. But, as we shall see later, unlike $K_0$, there is no need to unitalize for $L$. The following example shows the reason for that in the commutative case.

**Example 2.8.** Let $X$ be a compact Hausdorff space, $x \in X$, $Y = X\setminus\{x\}$, $A = C_0(Y)$, $A^+ = C(X)$. Let $[p]-[q] \in K_0(A)$, where $p, q \in M_n(A^+)$ are projections. Then $p = p_0+\alpha$, $q = p_0+\beta$, where $p_0$ is constant on $X$, and $\alpha, \beta \in M_n(A)$. Without loss of generality we may assume that $\alpha, \beta = 0$ not only at the point $x$, but also in a small neighborhood $U$ of $x$. Let $h \in C(X)$ satisfy $0 \le h \le 1$, $h(x) = 0$ and $h(z) = 1$ for any $z \in X\setminus U$. Set $a = hp_0+\alpha$, $b = hp_0+\beta$, then $a, b \in M_n(A)$ and $[(a,b)] \in L(A)$.

**Lemma 2.9.** *$L(\mathbb{C}) \cong \mathbb{Z}$.*

*Proof.* Let $a, b \in M_n$, $0 \leq a, b \leq 1$. Let $e_1, \ldots, e_n$ (resp. $e'_1, \ldots, e'_n$) be an orthonormal basis of eigenvectors for $a$ (resp. for $b$) with eigenvalues $\lambda_1, \ldots, \lambda_n$ (resp. $\lambda'_1, \ldots, \lambda'_n$). Let $0 < \lambda_i < 1$. Then $e_i$ is an eigenvector for $a - a^2$ with a non-zero eigenvalue $\lambda_i - \lambda_i^2$. As $(a - a^2)(a - b) = 0$, so $(a - b)(a - a^2) = 0$, hence

$$(a - b)(a - a^2)(e_i) = (\lambda_i - \lambda_i^2)(a - b)(e_i) = 0,$$

thus $(a - b)(e_i) = 0$, or, equivalently, $a(e_i) = b(e_i)$. As $e_i$ is an eigenvector for $a$, so it is an eigenvector for $b$ as well, $b(e_i) = \lambda_i e_i$. So, eigenvectors, corresponding to the eigenvalues $\neq 0, 1$, are the same for $a$ and $b$.

Re-order, if necessary, the eigenvalues so that

$$\lambda_1, \ldots, \lambda_k \in (0, 1), \qquad \lambda_{k+1}, \ldots, \lambda_n \in \{0, 1\},$$

and denote the linear span of $e_1, \ldots, e_k$ by $L$. Similarly, assume that

$$\lambda'_1, \ldots, \lambda'_{k'} \in (0, 1), \qquad \lambda'_{k'+1}, \ldots, \lambda'_n \in \{0, 1\},$$

and denote the linear span of $e'_1, \ldots, e'_{k'}$ by $L'$. As $e_1, \ldots, e_k \in L'$ and, symmetrically, $e'_1, \ldots, e'_{k'} \in L$, so $\dim L = \dim L'$, $k = k'$, and $\lambda_i = \lambda'_i$ for $i = 1, \ldots, k$.

Then $L^\perp$ is an invariant subspace for both $a$ and $b$, and the restrictions $a|_{L^\perp}$ and $b|_{L^\perp}$ are projections (as their eigenvalues equal 0 or 1). We may write $a$ and $b$ as matrices with respect to the decomposition $L \oplus L^\perp$:

$$a = \begin{pmatrix} c & 0 \\ 0 & p \end{pmatrix}; \qquad b = \begin{pmatrix} c & 0 \\ 0 & q \end{pmatrix}, \tag{4}$$

where $p, q$ are projections. The linear homotopy

$$a_t = \begin{pmatrix} tc & 0 \\ 0 & p \end{pmatrix}; \qquad b_t = \begin{pmatrix} tc & 0 \\ 0 & q \end{pmatrix}, \qquad t \in [0, 1],$$

connects the pair $(a, b)$ with the pair $(p, q) + (0, 0)$. Therefore, $L(\mathbb{C})$ is a quotient of $\mathbb{Z}$ (which is the set of homotopy classes of pairs of projections modulo stable equivalence). To see that $L(\mathbb{C})$ is exactly $\mathbb{Z}$, note that (4) implies that $\mathrm{tr}(a - b) \in \mathbb{Z}$ for any $a, b$ satisfying the relations (1), so this integer is homotopy invariant.

□

**Remark 2.10.** One may think that the relations (1) imply that $a, b$ are something like projections plus a common part and can be reduced to just a pair of projections by cutting out the common part. The following example shows that this is not that simple.

**Example 2.11.** Let $A = C(X)$, let $Y, Z$ be closed subsets in $X$ with $Y \cap Z = K$. Let $p, q \in M_n(C(Y))$ be projection-valued functions on $Y$ such that $p|_K = q|_K = r$, and let $r$ cannot be extended to a projection-valued function on $Z$ due to a $K$-theory obstruction, but can be extended to a matrix-valued function $s \in M_n(C(Z))$ on $Z$ (with $0 \leq s \leq 1$). Then set $a = \begin{cases} p & \text{on} \quad Y; \\ s & \text{on} \quad Z \end{cases}$ and $b = \begin{cases} q & \text{on} \quad Y; \\ s & \text{on} \quad Z \end{cases}$.

## 3. Universal $C^*$-algebra for relations (1)

Denote the $C^*$-algebra generated by $a, b$ satisfying (1) by $C^*(a, b)$. The universal $C^*$-algebra is the least $C^*$-algebra $D$ such that for any $a, b$ with (1) there is a surjective $*$-homomorphism $\varphi : D \to C^*(a, b)$, [5]. 'The least' means that for any surjective $*$-homomorphism $\psi : E \to C^*(a, b)$ there is a surjective $*$-homomorphism $\chi : E \to D$ such that $\psi = \varphi \circ \chi$.

Let $I \subset C^*(a,b)$ denote the ideal generated by $a-a^2$, and let $C^*(a,b)/I$ be the quotient $C^*$-algebra. Then $C^*(a,b)/I$ is generated by $\dot{a}=q(a)$ and $\dot{b}=q(b)$, where $q$ is the quotient map. But since $q(a-a^2)=q(b-b^2)=0$, $\dot{a}$ and $\dot{b}$ are projections, and $C^*(a,b)/I$ is generated by two projections.

Then the $C^*$-algebra $C^*(a,b)$ is completely determined by the ideal $I$, by the quotient $C^*(a,b)/I$, and by the Busby invariant $\tau : C^*(a,b)/I \to Q(I)$ (we denote by $M(I)$ the multiplier algebra of $I$ and by $Q(I)=M(I)/I$ the outer multiplier algebra). The latter is defined by the two projections $\tau(\dot{a}), \tau(\dot{b}) \in Q(C_0(Y))$, where $X=\mathrm{Sp}(a)$, $Y=X\setminus\{0,1\}$. Let $C_b(Y)$ denote the $C^*$-algebra of bounded continuous functions on $Y$ and let

$$\pi : C_b(Y) \to C_b(Y)/C_0(Y) = Q(C_0(Y))$$

be the quotient map. Using Gelfand duality, we identify $a$ with the function id on $\mathrm{Sp}(a)$. Let $f \in C_0(Y)$. Then

$$\tau(\dot{a})\pi(f(a)) = \tau(\dot{b})\pi(f(a)) = \pi(af(a)),$$

so we can easily calculate these two projections.

If $1 \notin X$ then $\tau(\dot{a})=\tau(\dot{b})=0$; if $X=\{1\}$ then $I=0$; if $1\in X$ and $X$ has at least one more point $x$ then $\tau(\dot{a})=\tau(\dot{b})$ is the class of functions $f$ on $X$ such that $f(1)=1$ and $f(t)=0$ for all $t\le x$.

Let $M_1 \subset M_2$ denote the upper left corner in the 2-by-2 matrix algebra. Set

$$D=\{f\in C([-1,1];M_2) : f(-1)=0, f(1) \text{ is diagonal}, f(t)\in M_1 \text{ for } t\in(-1,0]\}.$$

The structure of $D$ is similar to that of $C^*(a,b)$. The ideal

$$J=\{f\in D : f(t)=0 \text{ for } t\in[0,1]\} \cong C_0(-1,0)$$

is the universal $C^*$-algebra for $I$ (surjects on $I$ for any $0\le a\le 1$), and the quotient is the universal (nonunital) $C^*$-algebra

$$D/J = \mathbb{C}*\mathbb{C} = \{m\in C([0,1],M_2) : m(1) \text{ is diagonal}, m(0)\in M_1\} \tag{5}$$

generated by two projections [6]. Note that this $C^*$-algebra is an extension of $\mathbb{C}$ by the $C^*$-algebra $q\mathbb{C}=\{m\in C_0((0,1],M_2) : m(1) \text{ is diagonal}\}$ used in the Cuntz picture of $K$-theory.

**Lemma 3.1.** *The $C^*$-algebra $D$ is universal for the relations (1).*

*Proof.* For any $a,b$ satisfying (1) there are standard surjective $*$-homomorphisms $\alpha : J\to I$ and $\gamma : D/J \to C^*(a,b)/I$. Since $\alpha$ is surjective, it induces $*$-homomorphisms $M(\alpha): M(J)\to M(I)$ and $Q(\alpha): Q(J)\to Q(I)$ in a canonical way. As

$$D \cong \{(m,f) : m\in M(J), f\in D/J, q_J(m)=\tau(f)\},$$

$$C^*(a,b) \cong \{(n,g) : n\in M(I), g\in C^*(a,b)/I, q_I(n)=\sigma(g)\},$$

where $q_\bullet : M(\bullet)\to Q(\bullet)$ is the quotient map, so the map $\beta : D\to C^*(a,b)$ can be defined by $\beta(m,f)=(M(\alpha)(m),\gamma(f))$. This map is well defined if the diagram

$$\begin{tikzcd} D/J \arrow[r, "\tau"] \arrow[d, "\gamma"] & Q(J) \arrow[d, "Q(\alpha)"] \\ C^*(a,b)/I \arrow[r, "\sigma"] & Q(I) \end{tikzcd}$$

commutes. It does commute. The case $X=\mathrm{Sp}(a)=\{1\}$ is trivial. For other cases, notice that the image of $\tau$ lies in $C_0(0,1]/C_0(0,1)\subset Q(J)$, and the image of $\sigma$ lies in $C(X)/C_0(X\setminus\{0\})$, which is either $\mathbb{C}$ or $0$ (when $1\in X$ or $1\notin X$ respectively), and the

restriction of $Q(\alpha)$ from the image of $\tau$ to the image of $\sigma$ is induced by the inclusion $X \subset [0,1]$.

So, for any $A$ and any $a, b \in A$ satisfying (1) there is a surjective $*$-homomorphism from $D$ to $C^*(a,b)$. To see that $D$ is universal it suffices to show that $D$ is generated by some $\mathbf{a}, \mathbf{b}$ satisfying (1). Set

$$\mathbf{a}(t) = \begin{cases} \begin{pmatrix} \cos^2 \frac{\pi}{2}t & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [-1,0]; \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [0,1], \end{cases} \tag{6}$$

$$\mathbf{b}(t) = \begin{cases} \begin{pmatrix} \cos^2 \frac{\pi}{2}t & 0 \\ 0 & 0 \end{pmatrix} & \text{for } t \in [-1,0]; \\ \begin{pmatrix} \cos^2 \frac{\pi}{2}t & \cos \frac{\pi}{2}t \sin \frac{\pi}{2}t \\ \cos \frac{\pi}{2}t \sin \frac{\pi}{2}t & \sin^2 \frac{\pi}{2}t \end{pmatrix} & \text{for } t \in [0,1]. \end{cases} \tag{7}$$

Then $D$ is generated by these $\mathbf{a}$ and $\mathbf{b}$.

□

The $C^*$-algebra $D$ allows one more description. Set $A_0 = \mathbb{C}^2$, $F = \mathbb{C} \oplus M_2$ and define a $*$-homomorphism $\gamma : A_0 \to F \oplus F$ by $\gamma = \gamma_0 \oplus \gamma_1$, where $\gamma_0, \gamma_1 : \mathbb{C}^2 \to \mathbb{C} \oplus M_2$ are given by

$$\gamma_0(\lambda,\mu) = \lambda \oplus \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}; \quad \gamma_1(\lambda,\mu) = 0 \oplus \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}; \quad \lambda, \mu \in \mathbb{C}.$$

Let $\partial : C([0,1];F) \to F \oplus F$ be the boundary map, $\partial(f) = f(0) \oplus f(1)$, $f \in C([0,1];F)$. Then $D$ can be identified with the pullback

$$\begin{tikzcd} D = A_1 \arrow[r] \arrow[d] & A_0 \arrow[d, "\gamma"] \\ C([0,1];F) \arrow[r, "\partial"] & F \oplus F, \end{tikzcd}$$

$$D = \{(f,a) : f \in C([0,1];F), a \in A_0, \partial(f) = \gamma(a)\}.$$

Such pullback is called a 1-dimensional noncommutative CW complex (NCCW complex) in [4]; in this terminology, $A_0$ is a 0-dimensional NCCW complex.

Recall ([1]) that a $C^*$-algebra $B$ is *semiprojective* if, for any $C^*$-algebra $A$ and increasing chain of ideals $I_n \subset A$, $n \in \mathbb{N}$, with $I = \overline{\cup_n I_n}$ and for any $*$-homomorphism $\varphi : B \to A/I$ there exists $n$ and $\hat{\varphi} : B \to A/I_n$ such that $\varphi = q \circ \hat{\varphi}$, where $q : A/I_n \to A/I$ is the quotient map.

**Corollary 3.2.** *The $C^*$-algebra $D$ is semiprojective.*

*Proof.* Essentially, this is Theorem 6.2.2 of [4], where it is proved that all unital 1-dimensional NCCW complexes are semiprojective. The non-unital case is dealt in Theorem 3.15 of [7], where is it noted that if $A_1$ is a 1-dimensional NCCW complex then $A_1^+$ is a 1-dimensional NCCW as well, and semiprojectivity of $A_1$ is equivalent to semiprojectivity of $A_1^+$.

□

One more picture of $D$ can be given in terms of amalgamated free product: $D = C(0,1] *_{C_0(0,1)} C(0,1]$.

## 4. Identifying $L$ with $K_0$

Our definition of $L(A)$ can be reformulated in terms of the universal $C^*$-algebra $D$ as

$$L(A) = \varinjlim [D, M_n(A)],$$

where $[-,-]$ denotes the set of homotopy classes of $*$-homomorphisms. Recall that semiprojectivity is equivalent to stability of relations that determine $D$, (Theorem 14.1.4 of [5]). The latter means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $c, d \in A$ satisfy

$$\|c\| \leq 1, \quad \|d\| \leq 1, \quad c, d \geq 0, \quad \|(c - c^2)(c - d)\| < \delta, \quad \|(d - d^2)(c - d)\| < \delta,$$

there exist $a, b \in A$ such that $\|a - c\| < \varepsilon$, $\|b - d\| < \varepsilon$, and $a$, $b$ satisfy the relations (1). Stability of the relations (1) implies that

$$L(A) = [D, A \otimes \mathbb{K}] = [[D, A \otimes \mathbb{K}]],$$

where $\mathbb{K}$ denotes the $C^*$-algebra of compact operators, and $[[\cdot,\cdot]]$ is the set of homotopy classes of asymptotic homomorphisms.

**Lemma 4.1.** *The functor $L$ is half-exact.*

*Proof.* Let

$$0 \longrightarrow I \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be a short exact sequence of $C^*$-algebras. It is obvious that $p_* \circ i_* = 0$, so it remains to check that $\operatorname{Ker} p_* \subset \operatorname{Im} i_*$. Suppose that $a, b \in M_n(B)$ satisfy (1) and $(p(a), p(b)) = 0$ in $L(A)$. This means that there is a homotopy connecting $(p(a), p(b))$ to $(0, 0)$ in $M_k(A)$ for some $k \geq n$ such that the whole path satisfies (1). This homotopy is given by a $*$-homomorphism $\psi : D \to C([0,1], M_k(A))$ such that $\mathrm{ev}_1 \circ \psi = 0$, where $\mathrm{ev}_t$ denotes the evaluation map at $t \in [0,1]$.

When $D$ is a semiprojective $C^*$-algebra, the homotopy lifting theorem ([2], Theorem 5.1) asserts that, given a commuting diagram

$$\begin{tikzcd}
D \arrow[dr, dashed, "\varphi"'] \arrow[drr, "\kappa"] \arrow[ddr, "\psi"'] & & \\
& C([0,1]; M_k(B)) \arrow[r, "\mathrm{ev}_0"'] \arrow[d, "\overline{p}_k"] & M_k(B) \arrow[d, "p_k"] \\
& C([0,1]; M_k(A)) \arrow[r, "\mathrm{ev}_0"] & M_k(A),
\end{tikzcd}$$

where $\overline{p}_k$ and $p_k$ are the $*$-homomorphisms induced by a surjection $p$, there exists a $*$-homomorphism $\varphi$ completing the diagram. Replacing $A$ and $B$ by matrices over these $C^*$-algebras, we get a lifting $\varphi$ for the given homotopy. As $\mathrm{ev}_1 \circ \psi = 0$, so $\mathrm{ev}_1 \circ \varphi$ maps $D$ to $M_k(I)$. Thus the pair $(a, b)$ lies in the image of $i_*$.

$\square$

In a standard way, set $L_n(A) = L(S^n A)$, where $SA$ denotes the suspension over $A$. Then, by Theorem 21.4.3 of [3], $L_n(A)$, being homotopy invariant and half-exact, is a homology theory. Also, by Theorem 22.3.6 of [3] and by Lemma 2.9, it coinsides with the $K$-theory on the bootstrap category of $C^*$-algebras. We shall show now that it coinsides with the $K$-theory for any $C^*$-algebra.

Set

$$P = \begin{pmatrix} 1-\mathbf{b} & f(\mathbf{a}) \\ f(\mathbf{a}) & \mathbf{a} \end{pmatrix}; \quad Q = \begin{pmatrix} 1-\mathbf{b} & f(\mathbf{a}) \\ f(\mathbf{a}) & \mathbf{b} \end{pmatrix},$$

where $\mathbf{a}$, $\mathbf{b}$ are generators for $D$ ((6),(7)), and $f \in C_0(0,1)$ is given by $f(t) = (t-t^2)^{1/2}$. Then $P, Q \in M_2(D^+)$, where $D^+$ denotes the unitalization of $D$.

By Lemma 2.2, $f(\mathbf{a}) = f(\mathbf{b})$ and $\mathbf{a}f(\mathbf{a}) = \mathbf{b}f(\mathbf{a})$, so $P$ and $Q$ are projections. One also has $P - Q \in M_2(D)$, hence

$$x = [P] - [Q] \in K_0(D).$$

**Lemma 4.2.** *$K_0(D) \cong \mathbb{Z}$ with $x$ as a generator.*

*Proof.* Consider the short exact sequence

$$0 \longrightarrow J \longrightarrow D \xrightarrow{\pi} \mathbb{C} * \mathbb{C} \longrightarrow 0,$$

where $\mathbb{C}*\mathbb{C}$ is the universal (nonunital) $C^*$-algebra (5) generated by two projections, $p$ and $q$ [6], and $\pi$ is given by restriction to $[0,1]$, $\pi(\mathbf{a}) = p$, $\pi(\mathbf{b}) = q$. We have $\pi(P) = (1-q)\oplus p$, $\pi(Q) = (1-q) \oplus q$, so $\pi_*(x) = [p] - [q] \in K_0(\mathbb{C} * \mathbb{C})$. As $P(t) = Q(t)$ when $t \in [-1, 0]$, so for the boundary (exponential) map $\delta : K_0(\mathbb{C}*\mathbb{C}) \to K_1(J)$ we have $\delta(P) = \delta(Q)$. Recall that $J \cong C_0(-1, 0)$. Direct calculation shows that $\delta(P) = \delta(Q) \neq 0$. The claim follows now from the $K$-theory exact sequence

$$0 = K_0(J) \longrightarrow K_0(D) \xrightarrow{\pi_*} K_0(\mathbb{C} * \mathbb{C}) \xrightarrow{\delta} K_1(J) \cong \mathbb{Z}.$$

□

Let us define a map $\kappa : L(A) \to K_0(A)$. If $l = [(a, b)] \in L(A)$ then the pair $(a, b)$ determines a $*$-homomorphism $\varphi : D \to M_n(A)$ by $\varphi(\mathbf{a}) = a$; $\varphi(\mathbf{b}) = b$. So, $l \in L(A)$ determines a $*$-homomorphism $\varphi$ up to homotopy (for some $n$). Put

$$\kappa(l) = \varphi_*(x) \in K_0(A).$$

As this definition is homotopy invariant and as direct sum of pairs corresponds to direct sum of $*$-homomorphisms, so the map $\kappa$ is a well defined group homomorphism.

Recall that there is also a map $\iota : K_0(A) \to L(A)$ given by $\iota([p] - [q]) = [(p, q)]$, where $[p] - [q] \in K_0(A)$.

**Lemma 4.3.** *For any unital $C^*$-algebra $A$, one has $\kappa \circ \iota = \mathrm{id}_{K_0(A)}$; $\iota \circ \kappa = \mathrm{id}_{L(A)}$, hence $L(A) = K_0(A)$.*

*Proof.* To show the first identity, let $z \in K_0(A)$ and let $p, q \in M_n(A)$ be projections such that $z = [p] - [q]$. Let $\varphi : D \to M_n(A)$ be a $*$-homomorphism determined by the pair $(p, q)$. Then, due to universality of $\mathbb{C} * \mathbb{C}$, $\varphi$ factorizes through $\mathbb{C} * \mathbb{C}$, $\varphi = \psi \circ \pi$, where $\pi : D \to \mathbb{C}*\mathbb{C}$ is the quotient map and $\psi : \mathbb{C}*\mathbb{C} \to M_n(A)$ is determined by $\psi(i_1(1)) = p$ and $\psi(i_2(1)) = q$, where $i_1, i_2 : \mathbb{C} \to \mathbb{C} * \mathbb{C}$ are inclusions onto the first and the second copy of $\mathbb{C}$. Then

$$\varphi(x) = \psi_*([i_1(1)] - [i_2(1)]) = [p] - [q],$$

hence $\kappa(\iota(z)) = z$.

Let us show the second identity. For $[(a, b)] \in L(A)$, let $\varphi : D \to M_n(A)$ be a $*$-homomorphism defined by the pair $(a, b)$ (i.e. by $\varphi(\mathbf{a}) = a$, $\varphi(\mathbf{b}) = b$), and let $\varphi^+ : D^+ \to M_n(A)$ be its extension, $\varphi^+(1) = 1$. Then $\iota(\kappa([(a, b)])) = [(\varphi_2^+(P), \varphi_2^+(Q))]$, where $\varphi_2^+ = \varphi^+ \otimes \mathrm{id}_{M_2}$.

For $s \in [0, 1]$, set

$$P_s = C_s P C_s; \quad Q_s = C_s Q C_s, \qquad \text{where } C_s = \begin{pmatrix} s \cdot 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$P_s, Q_s \in M_2(D^+), \quad P_s - Q_s \in M_2(D), \quad 0 \leq P_s, Q_s \leq 1,$$
$$(P_s - P_s^2)(P_s - Q_s) = 0, \quad (Q_s - Q_s^2)(P_s - Q_s) = 0$$

for all $s \in [0,1]$; $P_0, Q_0 \in M_2(D)$, and

$$P_1 = P, \quad Q_1 = Q; \qquad P_0 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{a} \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{b} \end{pmatrix}.$$

Therefore, $(\varphi_2^+(P_s), \varphi_2^+(Q_s))$ provides a homotopy connecting $(\varphi_2^+(P), \varphi_2^+(Q))$ with $(0 \oplus a, 0 \oplus b)$, hence, the pair $(\varphi_2^+(P), \varphi_2^+(Q))$ is equivalent to the pair $(a, b)$.

□

**Theorem 4.4.** *The functors $L$ and $K_0$ coinside for any $C^*$-algebra $A$.*

*Proof.* Both functors are half-exact and coinside for unital $C^*$-algebras, so the claim follows.

□

**Remark 4.5.** Similarly to $D$, one can define a $C^*$-algebra $D_B$ for any $C^*$-algebra $B$ as an appropriate extension of $B * B$ by $CB$, where $CB$ is the cone over $B$ (or by $D_B = CB *_{SB} CB$). Then one gets the group $[D_B, A \otimes \mathbb{K}]$. Regretfully, $D_B$ has no nice presentation (unlike $D = D_{\mathbb{C}}$), so we don't pursue here the bivariant version.

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Moscow State University, Leninskie Gory, Moscow, 119991, Russia, and Harbin Institute of Technology, Harbin, P. R. China

*E-mail address*: manuilov@mech.math.msu.su